

The Eliminability of Delta in Goedel Logics¹

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Outline

Gödel logics: definitions

The Absoluteness Operator Δ

Restricted semantics

Chains and chain normal form

First-Order Limitations

History of Gödel Logics

- ▶ **1932–33 Gödel:** Introduced intermediate logics G_n .
Proved existence of infinitely many logics
between classical and intuitionistic.
- ▶ **1959 – Dummett’s Infinite-Valued Gödel Logic:** Replaced
finite chains by the full $[0, 1]$ interval, adding the
linearity axiom $(A \rightarrow B) \vee (B \rightarrow A)$;
- ▶ **1991 – Avron’s Hypersequent Calculus** with a communication
rule capturing linearity proof-theoretically.
- ▶ **1998 – Hájek’s t-Norm Logics:** Unified Gödel, Łukasiewicz,
and Product logics into the t-norm–based fuzzy-logic framework.
- ▶ **Since 1990s – Viennese School:** Baaz, Ciabattoni, Metcalfe,
Olivetti, Pichler, Zach *et al.* deepened proof theory, fragments,
Kripke semantics, and complexity analysis
- ▶ And many more...

Propositional logic

Usual propositional language, countably many propositional variables A_i , connectives $\wedge, \vee, \rightarrow$, constants $\perp, \top$.

Truth-value set $V \subseteq [0, 1]$, closed, with $0, 1 \in V$.

Evaluations

$$\{0, 1\} \subseteq V \subseteq [0, 1]$$

$$\mathcal{I}(A \wedge B) = \min\{\mathcal{I}(A), \mathcal{I}(B)\}$$

$$\mathcal{I}(A \vee B) = \max\{\mathcal{I}(A), \mathcal{I}(B)\}$$

$$\mathcal{I}(A \rightarrow B) = \begin{cases} \mathcal{I}(B) & \text{if } \mathcal{I}(A) > \mathcal{I}(B) \\ 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B). \end{cases}$$

Negation

Abbreviations $\neg A := A \rightarrow \perp$, $\top := \perp \rightarrow \perp$.

This yields the following definition of the semantics of $\neg$:

$$\mathcal{I}(\neg A) = \begin{cases} 0 & \text{if } \mathcal{I}(A) > 0 \\ 1 & \text{otherwise} \end{cases}$$

Why Define a Logic by Its Valid Sentences?

- ▶ Do not depend merely on the number of truth values.
- ▶ Depend on the order-topological structure of the truth-value set.
- ▶ The classification of first-order Gödel logics highly non-trivial.
- ▶ We *cannot* characterize a logic purely by its truth-value structure V (too many “representations”).

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$$\mathbf{G}_V = \{A : \forall \mathcal{I} \text{ into } V : \mathcal{I}(A) = 1\}$$

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Issue: Different $V \subseteq [0, 1]$ can induce the *same* G_V . e.g.

$$V_{\uparrow} = \{1 - \frac{1}{n} : n \geq 1\} \cup \{1\} \quad \text{and} \quad V'_{\uparrow} = \{1 - \frac{2}{n} : n \geq 1\} \cup \{1\}$$

both yield $G_{\uparrow}$.

Ensure a *one-to-one correspondence* between:

$$\{\text{distinct Gödel logics}\} \longleftrightarrow \{\text{sets of formulas } G_V\}.$$

Examples

$$V = \{0, 1\} \quad \rightarrow \mathbf{G}_V = CPL$$

$$V_1 = \{0, 1/2, 1\}, V_2 = \{0, 1/3, 1\} \quad \rightarrow \mathbf{G}_{V_1} = \mathbf{G}_{V_2}$$

$$V_{\uparrow} = \{1 - 1/n : n \geq 1\} \cup \{1\} \quad \rightarrow \mathbf{G}_{V_{\uparrow}} = \mathbf{G}_{\uparrow}$$

$$V_{\downarrow} = \{1/n : n \geq 1\} \cup \{0\} \quad \rightarrow \mathbf{G}_{V_{\downarrow}} = \mathbf{G}_{\downarrow}$$

$$V_m = \{1\} \cup \{1 - 1/k : 1 \leq k \leq m-1\} \rightarrow \mathbf{G}_{V_m} = \mathbf{G}_m$$

Strange Fact: there are *uncountably many* closed subsets of $[0, 1]$ but only *countably many* distinct propositional Gödel logics.

Baaz Δ operator

“Is A absolutely true?”

$$\mathcal{I}(\Delta A) = \begin{cases} 1 & \mathcal{I}(A) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- ▶ while $\neg\neg A$ detects whether $A > 0$, there is no connective that detects whether $A < 1$.
- ▶ Δ allows the definition of a strict order $<_{\Delta}$.

$$A <_{\Delta} B \equiv \neg\Delta(B \rightarrow A)$$

Proposition

Δ is not definable with other connectives ($\wedge, \vee, \rightarrow, \perp, \top$) and variables

Proof.

There are a finite number of 1-variable functions in Gödel logic $\top, \perp, A, \neg A, \neg A \vee A, \neg A \rightarrow A$.

Let $\Delta(A) \leftrightarrow F(A)$ for some function F .

Look at the F in G_3 : Δ not definable in $G_3 \Rightarrow$ not definable in all larger Propositional Gödel logics.

A	$\neg A$	$\top$	$\perp$	$A \vee \neg A$	$\neg A \rightarrow A$	ΔA
0	1	1	0	1	0	0
1/2	1/2	1	0	1/2	1	0
1	0	1	0	1	1	1

none of them defines Δ and they are closed under composition by all connectives. □

Axiomatisation of Δ by Baaz

Hilbert style axiomatisation of the Δ operator (semantically a crisping operator - behaves S4-like):

$$\Delta 1 \quad \Delta A \vee \neg \Delta A$$

$$\Delta 2 \quad \Delta(A \vee B) \rightarrow (\Delta A \vee \Delta B)$$

$$\Delta 3 \quad \Delta A \rightarrow A$$

$$\Delta 4 \quad \Delta A \rightarrow \Delta \Delta A$$

$$\Delta 5 \quad \Delta(A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B)$$

$$A \vdash \Delta A$$

Then $G_{[0,1]}^\Delta$ is axiomatized by $G_{[0,1]} + \Delta$.

Δ is useful, but problematic

Why Δ is useful

- ▶ detects absolute truth;
- ▶ restores the missing symmetry at 1;
- ▶ gives extra expressive power;
- ▶ has natural modal/stability readings.

Why Δ is difficult

- ▶ not definable by standard connectives;
 - ▶ breaks substitution closure;
 - ▶ affects the deduction theorem;
 - ▶ complicates proof theory;
 - ▶ equivalence principle fails;
 - ▶ is not intermediate logic anymore.
-
- ▶ Over the full interval $[0, 1]$ the logic admits hypersequent calculi; for the first-order case - even cut-free hypersequent calculi

Why Eliminate Δ ?

- ▶ Simplifies semantic and proof-theoretic analysis
- ▶ Restores substitution-closure (lost once Δ is added)
- ▶ Influences axiomatizations and decision procedures

The problem

Can we eliminate Δ ?

More precisely:

- ▶ Can every formula with Δ be transformed into a formula without Δ ?
- ▶ Does the transformation preserve validity?
- ▶ Does it preserve logical equivalence?
- ▶ What changes in the first-order case?

This motivated our recent paper

(Matthias Baaz, Mariami Gamsakhurdia:) **Goedel Logics: On the Elimination of The Absoluteness Operator-** TBiLLC2025
Postproceedings.

Two notions of elimination

1. Validity-equivalent elimination

$$\models A \quad \text{iff} \quad \models A^T,$$

where A^T is Δ -free.

This is enough for decision procedures and proof search.

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where A^T is Δ -free.

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2. Logically-equivalent elimination

$$A \equiv A^T.$$

This is much stronger: the two formulas have the same truth value under every interpretation.

Two Ingredients for Elimination

1. Restricted Semantics
2. Chain Normal Forms

Algorithm:

- ▶ Restrict the semantics,
- ▶ Systematically transform formulas containing $\triangle$ into chain normal forms,
- ▶ Decompose complex expressions into linear chains without $\triangle$.

Restricted semantics G_V^-

Instead of attacking Δ directly, we slightly restrict the semantics.

Atomic formulas are never allowed to take value 1.

Only the truth constant $\top$ may have value 1.

We denote the corresponding logic by G_V^- .

- ▶ The notion, to our knowledge, is novel and has not been systematically studied in the literature.

Theorem: Reduction to standard semantics

Restricted semantics can be encoded inside standard semantics with Δ .

For a propositional formula $F(X_1, \dots, X_n)$:

F is valid under restricted semantics

iff

$$(\neg \Delta X_1 \wedge \dots \wedge \neg \Delta X_n) \rightarrow F$$

is valid under standard semantics with Δ .

The antecedent simply says:

none of the variables has value 1.

Uniform Substitution failure

- ▶ When propositional variables are replaced by arbitrary formulas:
As $\neg\Delta p$ is true for any variable p , substituting p by $\top$ yields $\neg\Delta\top$ being false.
- ▶ Nevertheless, restricted validity is preserved under substitutions σ satisfying

$$\mathcal{I}(\sigma(p)) < 1$$

for every $\mathcal{I}$ and p in the domain of σ .

- ▶ Indeed, such substitutions induce restricted valuations, so the usual substitution lemma applies - restricted validity is preserved under renaming of propositional variables.

This failure of uniform substitution does not imply a failure of replacement of semantically equivalent formulas.

Equivalence Principle

$$A \leftrightarrow B \Rightarrow E(A) \leftrightarrow E(B)$$

The specific case

$$A \leftrightarrow B \Rightarrow \Delta(A) \leftrightarrow \Delta(B)$$

for Δ -operator fails also in the restricted semantics.

Example

$$\mathcal{I}(A) = 1, \quad 0 < \mathcal{I}(B) < 1$$

$$\mathcal{I}(\Delta A) = 1, \quad \mathcal{I}(\Delta B) = 0$$

$\mathcal{I}(A \leftrightarrow B)$ is not 0.

Lemma (Context-Closure)

For any formulas A, B, C, D , and any context function E , the following implication holds: from $A \vee B \wedge (C \leftrightarrow D)$ we derive $A \vee B \wedge (E(C) \leftrightarrow E(D))$

Lemma (Transitivity)

The transitive closure of equivalence in the restricted semantics holds, i.e., for any formulas C, D, L :

$$(A \vee (B \wedge (C \leftrightarrow D))) \wedge (D \leftrightarrow L) \Rightarrow (A \vee B \wedge (C \leftrightarrow L)).$$

Lemma (Local Δ -Elimination)

Given any expression $\Delta(C \vee (D \wedge a))$ for some variable a in the restricted semantics where C, D are valid expressions we can eliminate Δ obtaining C

$$\frac{\Delta(C \vee (D \wedge a))}{C}$$

Chains replace valuations by order types

Instead of considering infinitely many truth values, we consider finitely many possible order patterns among the variables.

Example:

$$\perp < a < b < \top,$$

$$\perp < b < a < \top,$$

$$\perp < a \leftrightarrow b < \top.$$

The actual values do not matter as much as their ordering.

- ▶ Every chain describes an order type of the variables X .
- ▶ For each valuation there is a chain that matches the ordering of truth-values assigned to atoms.

Chains: the proof-theoretic intuition

Every Gödel valuation induces a linear ordering of truth values.

A **chain** is a formula describing one such ordering relative to the constants $\top$ and $\perp$, described by positions of variables.

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A **chain** C over the set of propositional variables $X = [X_1, \dots, X_n]$ is an expression

$$\perp \triangleright_1 X_{\Pi(1)} \triangleright_2 \top \triangleright_1 X_{\Pi(2)} \triangleright_3 \cdots \triangleright_n X_{\Pi(n)} \triangleright_{n+1} \top$$

Π is permutation on $[1 \dots n]$ and $\triangleright_i \in \{<, \leftrightarrow\}$.

Chains play for Gödel logic a role similar to **disjunctive normal forms** in classical logic.

Restricted chains

In restricted semantics, no variable may be equivalent to $\top$.

So we allow chains such as

$$\perp < a < b < \top,$$

but not chains such as

$$\perp < a < b \leftrightarrow \top.$$

We denote a chain in the restricted semantics by C^- .

This small change is what makes **logically-equivalent elimination** possible at the **propositional** level.

Main (Propositional) Result

Every propositional formula containing Δ is equivalent, under restricted semantics, to a disjunction of Δ -free restricted chains.

Thus Δ is eliminable propositionally under restricted semantics.

Example: the intuition

Consider a formula of the form

$$F := a \vee \Delta(a \vee \neg a).$$

The corresponding chain decomposition yields three chains in standard semantics:

$$(\perp \leftrightarrow a) < \top, (\perp < a) < \top, (\perp < \top \leftrightarrow a)$$

for some variable a .

Example: the intuition

Consider a formula of the form

$$F := a \vee \Delta(a \vee \neg a).$$

The corresponding chain decomposition yields three chains in standard semantics:

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for some variable a .

Under restricted semantics, the relevant chains are only:

$$\perp \leftrightarrow a < \top, \quad \perp < a < \top.$$

The chain where $a \leftrightarrow \top$ is excluded.

The CNF in the restricted semantics

$$(\perp \leftrightarrow a) < \top \wedge F \vee (\perp < a) < \top \wedge F.$$

evaluate from the innermost first.

Evaluation of the first chain:

$$a \vee \Delta(a \vee \top)$$

$$a \vee \Delta \top$$

$$a \vee \top$$

$$\top$$

Evaluation of the second chain:

$$a \vee \Delta(a \vee \perp)$$

$$a \vee \Delta a$$

$$a \vee \perp$$

$$a$$

$$\perp$$

After evaluation along the restricted chains, the formula becomes Δ -free.

$$a \vee \Delta(a \vee a \rightarrow \perp) \leftrightarrow \neg a.$$

Standard vs restricted semantics

Standard semantics	Restricted semantics
Atoms may be 1	Atoms are always < 1
Δ detects atomic truth	$\Delta P = 0$ for atoms P
Substitution closure holds without Δ	Uniform substitution closure fails
Δ not logically eliminable	Δ logically eliminable propositionally

Without Δ : standard and restricted coincide

For propositional formulas without Δ :

$$\mathbf{G}_{[0,1]} \models A \quad \text{iff} \quad \mathbf{G}_{[0,1]}^- \models A.$$

Why?

Without Δ , formulas cannot distinguish exact truth from values arbitrarily close to truth.

So the restriction only matters when Δ is present.

First-Order Limitations

Full characterization of Axiomatizability

Theorem (Baaz, Preining)

A first-order Gödel logic G_V is recursively enumerable iff one of the following conditions is satisfied:

- 1. V is finite,*
- 2. V is uncountable and 0 is an isolated point,*
- 3. V is uncountable, and every neighbourhood of 0 is in the perfect subset.*

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Not recursively enumerable

- ▶ countably infinite truth value set
- ▶ every neighbourhood of 0 is countably infinite

Full characterization of Δ -Axiomatizability

Recursively axiomatizable

- ▶ finitely valued
- ▶ V is uncountable and 0 is an isolated point, and 1 is in the perfect subset,
- ▶ V is uncountable, and 0 is in the perfect subset, and 1 is isolated point,
- ▶ V is uncountable and both 0 and 1 are isolated points,
- ▶ V is uncountable, and 0 and 1 is in the perfect subset.

Full characterization of Δ -Axiomatizability

Recursively axiomatizable

- ▶ finitely valued
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Not recursively enumerable

- ▶ countably infinite truth value set
- ▶ either 0 or 1 is not isolated but not in the perfect kernel

First-Order Limitations

The argument used for the propositional case does not extend to the first-order case.

- ▶ Case: 1 is not isolated, not in a perfect set.
0 is isolated or in a perfect set $\Rightarrow$
the first-order $\mathbf{G}_V^\Delta$ is not recursively enumerable,
while the first-order $\mathbf{G}_V$ is.
This holds both for standard and restricted semantics
- ▶ Therefore, there is not even an effective validity equivalence
elimination of Δ , $\Rightarrow$ no valid equivalence as in the propositional
case.

What changes in first-order logic?

In first-order Gödel logic, quantifiers are interpreted by infima and suprema:

$$I(\forall x A(x)) = \inf_u I(A(u)),$$

$$I(\exists x A(x)) = \sup_u I(A(u)).$$

This creates new problems.

The value 1 may be approached without ever being attained.

New Ingredient: Witnessed semantics

Witnessed semantics removes this pathology.

Instead of arbitrary infima and suprema, we require extrema to be attained:

$$I(\forall x A(x)) = \min_u I(A(u)),$$

$$I(\exists x A(x)) = \max_u I(A(u)).$$

Then existential truth has an actual witness.

This improves the interaction between quantifiers and Δ .

Infimum is attained: $\exists x (A(x) \rightarrow \forall y A(y))$,

Supremum is attained: $\exists x (\exists y A(y) \rightarrow A(x))$.

$$(\forall x A(x) \rightarrow B) \rightarrow \exists x (A(x) \rightarrow B) \quad (A \rightarrow \exists x B(x)) \rightarrow \exists x (A \rightarrow B(x)).$$

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$$(\forall x A(x) \rightarrow B) \rightarrow \exists x (A(x) \rightarrow B) \quad (A \rightarrow \exists x B(x)) \rightarrow \exists x (A \rightarrow B(x)).$$

$$\Delta \forall x A(x) \leftrightarrow \forall x \Delta A(x)$$

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$$\Delta \forall x A(x) \leftrightarrow \forall x \Delta A(x)$$

In contrast, the existential quantifier exhibits an asymmetry:

$$\exists x \Delta A(x) \rightarrow \Delta \exists x A(x),$$

Converse implication fails in general.

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In contrast, the existential quantifier exhibits an asymmetry:

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$GW_{[0,1]}^\Delta$ is axiomatized by

$$GW_{[0,1]} + \Delta + \exists x \Delta A(x) \leftrightarrow \Delta \exists x A(x).$$

Delta Elimination

1. Standard + Witnessed Semantics: Effective Validity-equivalent Elimination of Δ
by an effective translation $A \mapsto A^T$ from formulas containing Δ to formulas without Δ such that $GW_V^\Delta \models A$ iff $GW_V \models A^T$ [3].
2. Restricted + Witnessed Semantics: Logically-Equivalent elimination of Δ - as full collection of quantifier-shift principles is available and axiomatized.
Move all quantifiers in prenex form and eliminate occurrences of Δ from the quantifier-free matrix of a formula via elementary equivalences as before.

Conceptual conclusion

Setting	Behavior of Δ
Propositional, standard	not logically eliminable
Propositional, restricted	logically eliminable
First-order, standard	generally not eliminable
First-order, witnessed	validity-equivalent elimination
Restricted + witnessed	stronger elimination behavior

- Under restricted semantics every propositional formula with Δ has a Δ -free equivalent (disjunction of chains).

Conclusions & Open Questions

- ▶ Extend to first-order logic when 1 is isolated
- ▶ Can the first-order restricted case be fully classified?
- ▶ Are similar restrictions useful for other intermediate or modal logics?

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Conjecture: In the restricted semantics all uncountable infinite-valued G_V^- coincide with $G_{[0,1]}^-$ and all countable witnessed infinite-valued G_V^- are not r.e.

In G_V where 1 is not isolated, not in perfect set and 0 is in the perfect set, there is no effective validity-equivalence translation between $G_V^{\Delta-}$ and G_V^- .

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Thank You!

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